Step frames analysis
(in single- and multi-conclusion calculi)

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20 Years in Free Modal Algebras Constructions

Step-by-step constructions of free modal algebras have longstanding tradition:

- they can traced back to Fine normal forms (1975);
- Abramsky (1988, published much later in 2005);
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Coumans-van Gool emphasis is on partial algebras; a light reformulation of their point of view uses *two-sorted structures*.

These are called step-algebras and step-frames in (N. Bezhanishvili-Ghilardi-Jibladze, in press).

We review step-algebras and step-frames and then discuss ongoing work *concerning proof theoretic applications*

We outline the kind of proof theoretic aspects we want to investigate.

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A logic L is a set of formulae containing tautologies, Aristotle's law and closed under necessitation, modus ponens and uniform substitution.

An inference system Ax for L is a set of inference rules

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- all rules in *Ax* are derivable from *L*;
- from the rules in *Ax* all formulae in *L* are provable;
- all formulae occurring as premises or as conclusions of rules in *Ax* have modal degree at most 1.

Rules satisfying the last condition are called *reduced*; non-reduced rules can be equivalently (for our purposes) replaced by reduced ones.

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... not all equivalent inference systems are proof-theoretically equally good ...

... for some of them proof search may become intricated ...

... we want at least the following: *to prove a formula ϕ, only formulae up to the modal degree of ϕ are needed*.

Since the above property leads to decidability, it won't be possible to get it in general, but we want to have some criteria to recognize good systems.

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As an example, take GL-system. It can be axiomatized by the single axiom $\Box(\Box x \to x) \to \Box x$.
An inference system for L consists of transitivity and Gödel-Löb rules

$$\frac{\Box^+ y \to x}{\Box y \to \Box x} \qquad \frac{\Box x \to x}{x} \tag{1}$$

An alternative one (the good one!) consists of the rule

$$\frac{\Box^+ x \wedge \Box y \to y}{\Box x \to \Box y} \tag{2}$$

Logics and Inference Systems

Given a logic L, for a finite set of formulae Γ and for a formula ϕ, we write

$$\Gamma \vdash_L \phi \tag{3}$$

for the *global* consequence relation: this means that ϕ has a proof from axioms and rules of L with premises Γ (uniform substitution applies only to formulae in L, not to formulae in Γ). For transitive systems, $\Gamma \vdash_L \phi$ is the same as $\Box^+(\bigwedge \Gamma) \to \phi \in L$.
For axioms systems Ax, we have a similar notion

$$\Gamma \vdash_{Ax} \phi \tag{4}$$

Notice that if Ax is an inference system for L, the relations (3) and (4) are equivalent.

Logics and Inference Systems

Definition

An inference system Ax has the *bounded proof property* (bpp) iff whenever $\Gamma \vdash_{Ax} \phi$ holds, there is a proof in Ax of ϕ with premises Γ in which formulae not exceeding the modal degree of formulae in Γ, ϕ occur.

Definition

A logic L has the *finite model property* (fmp) iff whenever $\Gamma \vdash_{Ax} \phi$ does not hold, then there is a Kripke model based on a finite frame for L where the Γ are everywhere true and ϕ is not.

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Outline

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1. Ordinary Rules
   - Step algebras and step frames
   - The Step Embedding Theorem
   - An Example
   - Some Case Studies

2. Multiconclusion Rules
   - A Hilbert calculus for hyperformulae
   - Step Frame Characterizations
   - Stable Classes

One-Step Modal Algebras

Definition

1. A *one-step modal algebra* is a quadruple $(A_0, A_1, i_0, \Diamond_0)$, where A_0, A_1 are Boolean algebras, $i_0 : A_0 \to A_1$ is a Boolean morphism, and $\Diamond_0 : A_0 \to A_1$ is a semilattice morphism. The algebras A_0, A_1 are called the *source* and the *target* Boolean algebras of the one-step modal algebra $(A_0, A_1, i_0, \Diamond_0)$.
2. A *one-step extension* of the one-step modal algebra $(A_0, A_1, i_0, \Diamond_0)$ is a one-step modal algebra $(A_1, A_2, i_1, \Diamond_1)$ satisfying $i_1 \Diamond_0 = \Diamond_1 i_0$.

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Applying Duality

To get a better understanding of the situation, it is nice to apply duality. Since we are interested in finite algebras, duality needs not topological machinery and is easy.

Definition

1. A *one-step frame* is a quadruple (W_1, W_0, f, R), where W_0, W_1 are sets, $f : W_1 \longrightarrow W_0$ is a function and $R \subseteq W_1 \times W_0$ is a relation.
2. A *one-step extension* of the one-step frame (W_1, W_0, f, R) is a one-step frame (W_2, W_1, g, S) satisfying $f \circ S = g \circ R$.

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Step Correspondence Theory

We need to define what it means for for (the dual of) a one-step frame (W_1, W_0, f, R) to validate an inference rule. The definition is almost straightforward; we illustrate it via the example of the **K4** rule

$$\frac{\Box^+ y \to x}{\Box y \to \Box x}$$

This happens iff

$$\forall x, y \subseteq W_0 \; (f^*(x) \cap \Box_R x \subseteq f^*(y) \Rightarrow \Box_R x \subseteq \Box_R y). \tag{5}$$

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Step Correspondence Theory

Applying adjunction $\exists_f \dashv f^*$, Ackermann rule, set-theoretic definition of $\subseteq$, definition of $\Box_R$ and Ackermann rule again we get:

$$
\begin{array}{rc}
\forall x, y \subseteq W_0. & f^*(x) \cap \Box_R x \subseteq f^*(y) \Rightarrow \Box_R x \subseteq \Box_R y \\
\forall x, y \subseteq W_0. & \exists_f (f^*(x) \cap \Box_R x) \subseteq y \Rightarrow \Box_R x \subseteq \Box_R y \\
\forall x \subseteq W_0. & \Box_R x \subseteq \Box_R \exists_f \, (f^*(x) \cap \Box_R x) \\
\forall x \subseteq W_0. & \Box_R x \subseteq \Box_R \exists_f \, (\Box_R x) \\
\forall x \subseteq W_0 \forall w \in W_0. & w \in \Box_R x \Rightarrow w \in \Box_R \exists_f \, (\Box_R x). \\
\forall x \subseteq W_0 \; \forall w \in W_0. & R(w) \subseteq x \Rightarrow w \in \Box_R \exists_f \, (\Box_R x) \\
\forall w \in W_0. & w \in \Box_R \exists_f \, (\Box_R R(w)).
\end{array}
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Last condition reads as:

$$\forall w \forall v \, (R(w, v) \to \exists w_1 \, (f(w_1) = v \,\&\, R(w_1) \subseteq R(w))) \, . \qquad (6)$$

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The Step Embedding Theorem

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A one-step frame (W_1, W_0, f, R) is *conservative* iff (i) f is surjective and (ii) for every $w_1, w_2 \in W_1$ we have that

$$f(w_1) = f(w_2) \ \& \ R(w_1) = R(w_2) \ \Rightarrow \ w_1 = w_2. \tag{7}$$

Dually, a one-step modal algebra $(A_0, A_1, i, \Diamond)$ is conservative iff (i) i is injective and (ii) the set

$$\{i(a) \mid a \in A_0\} \cup \{\Diamond a \mid a \in A_0\}$$

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The Step Embedding Theorem

Theorem

Let Ax be an inference system for a logic L. Then every finite conservative step-frame validating Ax is a p-morphic image of a finite Kripke frame for L iff Ax has the bpp and L has the fmp.

Kripke frames are here seen as step-frames with identical step transition function f; the notion of a p-morphism between step-frames is the obvious one.

The Step Embedding Theorem

A *p-morphism* between step frames $\mathcal{F}' = (W_1', W_0', f', R')$ and $\mathcal{F} = (W_1, W_0, f, R)$ is a pair of surjective maps $\mu : W_1' \longrightarrow W_1, \quad \nu : W_0' \longrightarrow W_0$ such that

$$f \circ \mu = \nu \circ f' \quad \text{and} \quad R \circ \mu = \nu \circ R' \ . \tag{8}$$

$$\begin{tikzcd} W_1' \arrow[r, "\mu"] \arrow[d, "f'"'] & W_1 \arrow[d, "f"] \\ W_0' \arrow[r, "\nu"] & W_0 \end{tikzcd} \qquad \begin{tikzcd} W_1' \arrow[r, "\mu"] \arrow[d, "R'"'] & W_1 \arrow[d, "R"] \\ W_0' \arrow[r, "\nu"] & W_0 \end{tikzcd}$$

An example

Using the above theorem, it is easy to prove fmp and bpp for simple logics like $K, K4, T, S4, \ldots$. As an example, let us consider the density axiom $\Box\Box x \to \Box x$ (we add this axiom to K).

First, we turn the axiom into a rule; there is a default naif method for that giving

$$\frac{y \to \Box x}{\Box y \to \Box x}. \tag{9}$$

Second, applying step correspondence, we get the following first-order characterization for validation of (9) in step frames:

$$\forall w \forall v\ (wRv \Rightarrow \exists k\ (wRf(k)\ \&\ kRv)) \tag{10}$$

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Third, we fix a finite conservative step-frame $\mathcal{S} = (W_1, W_0, f, R)$ satisfying (10); we must find a finite frame $\mathfrak{F} = (V, S)$ which dense in the standard sense

$$\forall w \forall v\ (wSv \Rightarrow \exists k\ (kSv\ \&\ wSk)). \tag{11}$$

and a surjective map $\mu : V \longrightarrow W_1$ such that $R \circ \mu = f \circ \mu \circ S$.

The idea is to take $V := W_1$ and $\mu := id_{W_1}$, so that we need to check

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Thus, taking into consideration that f is also surjective (because S is conservative)

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can be shown to be valid manually or by a prover (SPASS solves the problem in less than half a second; the superposition proof it finds takes 47 lines).

As a consequence of this we get altogether bpp, fmp and first order definability (decidability and canonicity also follows by general reasons).

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$$(14)\ \&\ (10)\ \&\ (13)\ \Rightarrow\ (12)\ \&\ (11).$$

can be shown to be valid manually or by a prover (SPASS solves the problem in less than half a second; the superposition proof it finds takes 47 lines).

As a consequence of this we get altogether bpp, fmp and first order definability (decidability and canonicity also follows by general reasons).

An example

Let's summarize the three steps:

- First: produce the inference rules (there are automatic methods, not always they give the good rules).
- Second: apply correspondence theory (this is automatic).
- Third: produce p-morphic extensions to standard frames (not automatic, but there are templates); provers can discharge the final proof obbligation.

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Some case studies

We wonder to which extent the above mechanization of the metatheory can be pushed.

We analyzed some more significant cases. The first case is **GL** system axiomatized by the single axiom $\Box(\Box x \to x) \to \Box x$.

First Step can be driven so that to obtain a rule which is equivalent (for our purposes) to the well-known rule

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Second Step (via Ackermann rule applied to fixpoint logic) gives

$$\forall w\; R(w) \subseteq \mu(Y, w)\exists_f(f^*(R(w)) \cap \Box_R R(w) \cap \Box_R Y). \tag{15}$$

In *finite* one-step frames this simplifies to

$$\forall w\; (R(w) \subseteq \{f(w') \mid R(w') \subset R(w)\})\,. \tag{16}$$

Notice that $\subset$ is strict inclusion, so the above condition is a 'step' irreflexivity.

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Third Step is not difficult, but is not fully automatic. We can use the template S for transitive systems, but then the resulting Kripke frame is not irreflexive, so one needs to take the dijoint union of the irreflexive subframes satisfying (12).

It should be noticed that, if we do the same analysis for the system axiomatized by transitivity and Löb rule, we get a weaker condition than (15). Using the fact that the condition is too weak, it is possible to prove formally that bpp fails.

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Our second case study is the system **S4.3** axiomatized via **S4** reflexivity and transitivity axioms plus $\Box(\Box x \to y) \vee \Box(\Box y \to x)$.

First step: the inference rule extracted automatically from the axiom is not good (bpp fails). Instead, we use Goré infinitely many rules:

$$\frac{\cdots \Box y \to x_j \vee \bigvee_{j \neq i} \Box x_i \cdots}{\Box y \to \bigvee_{i=1}^{n} \Box x_i} \tag{17}$$

The rules are indexed by n and the n-th rule has n premises, according to the values $j = 1, \ldots, n$.

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Second Step: correspondence theory applies to these rules. Interpreting the results in finite frames one gets

$$\forall w\, \forall S \subseteq R(w)\, \exists v \in S\, \exists w'\, (f(w') = v \;\&\; S \subseteq R(w') \subseteq R(w)). \qquad (18)$$

Third Step: the same method used in **GL** case shows that one-step frames satisfying (18) are p-morphic images of Kripke frames for **S4.3**. This establishes bpp and fmp.

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First Step The following rule has been proposed in the literature:

$$\frac{\Box\Gamma \Rightarrow y, \Box\Delta}{\Box\Gamma \Rightarrow \Box y, \Box\Delta} . \tag{19}$$

In the resulting system, cuts cannot be completely eliminated, but can be limited to subformulae of the sequent to be proved. This 'analytic' cut-elimination property is sufficient to imply the bpp, and thus we should be able to get the bpp directly by our methods.

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Second Step Correspondence theory gives

$$\forall w \forall v \ \ (wRv \to \exists \tilde{w} \ (f(\tilde{w}) = v \ \& \ R(w) = R(\tilde{w}))). \tag{20}$$

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As a final case study consider the system obtained by adding to **K** the axiom $\Box\Box x \leftrightarrow \Box x$. This is density+transitivity; we can join the rules we already used for density and transitivity. This is not a good idea: bpp fails!

Instead, we use the following couple of rules suggested to us by G. Mints:

$$\frac{\Box^+\Gamma \to \alpha}{\Box\Gamma \to \Box\alpha} \qquad \frac{\Gamma, \Box\Delta \Rightarrow \Box\alpha}{\Box\Gamma, \Box\Delta \Rightarrow \Box\alpha} \tag{21}$$

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Third Step Step frames satisfying the above property are easily seen to be p-morphic images of transitive and dense Kripke frames; this establishes bpp and fmp.

Notice that the fact that (10)+ (6) do not imply (22) is a formal argument proving that bpp fails is we adopt the old rule (9) in a transitive context. Thus, at least in principle, *model finders* can be used as automatic supports for showing that bpp fails.

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Why many conclusions?

A *multiple-conclusion rule* is a pair of finite sets of formulae $\langle \Gamma, S \rangle$.

If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}, S = \{\delta_1, \ldots, \delta_m\}$, we write the rule $\langle \Gamma, S \rangle$ as Γ / S or as

$$\frac{\gamma_1, \ \cdots, \ \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

The formulae $\Gamma = \{\gamma_1, \ldots . \gamma_n\}$ are said to be the *premises* of the rule (R) and the formulae $S = \{\delta_1, \ldots, \delta_m\}$ are said to be the *conclusions* of the rule (R).

The rule (R) is *valid* in a modal algebra $(A, \Box)$ iff for every valuation V

$$V(\gamma_1) = 1 \ \& \ \cdots \ \& \ V(\gamma_n) = 1 \quad \Rightarrow \quad V(\delta_1) = 1 \text{ or } \cdots \text{ or } V(\delta_m) = 1 \ .$$

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Multiple-conclusion rules recently gained attention in the literature from many points of view.

From an algebraic and a semantic point of view, (Kracht 07, Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), they constitute an essential tool for investigating classes of algebras beyond varieties and they supply nice canonical formulae axiomatizations.

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Compare e.g. the simplicity of the hypersequent rule

$$\frac{\tilde{\Gamma}, \Box\Gamma, \Box\Gamma' \Rightarrow \Delta \qquad \tilde{\Gamma}', \Box\Gamma', \Box\Gamma \Rightarrow \Delta'}{\tilde{\Gamma}, \Box\Gamma \Rightarrow \Delta \mid \tilde{\Gamma}', \Box\Gamma' \Rightarrow \Delta'} \ (\mathit{Dich})$$

for **S4**.**3** with the above Goré rules.
Rule (Dich) can be rewritten as multiconclusion rule to

$$\frac{\Box\gamma \wedge \Box\gamma' \to \delta \qquad \Box\gamma' \wedge \Box\gamma \to \delta'}{\Box\gamma \to \delta \mid \Box\gamma' \to \delta'}$$

Notice however that (Dich) does not define the variety of **S4**.**3** algebras but a universal class of algebras

$$\forall x\, \forall y\, (\Box x \leq \Box y \text{ or } \Box y \leq \Box x)$$

generating it.

Derived Rules

Let K be a set of multiple-conclusion rules; a multiple-conclusion rule Γ/S is *derivable from K* - written $K \vdash \Gamma/S$ iff every modal algebra validating all rules in K validates also Γ/S.

In the terminology of modal rule systems (Jerabek 09, N. & G. Bezhanishvili & Iemhoff 2014), it can be proved that this equivalently means that Γ/S belongs to the smallest modal rule system including K.

What we want to build here is a *Hilbert style* calculus for recognizing $K \vdash \Gamma/S$. This calculus will consequently be complete also for global consequence relation in K.

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Hyperformulae and derivations

Our calculus will manipulate hyperformulae, seen as disjunctions of global assertions (this is the shape of conclusions of our multi-conclusion rules).

A *hyperformula* is a finite set of propositional formulae written in the form

$$\alpha_1 \mid \cdots \mid \alpha_n. \tag{23}$$

We use letters $S, S_1, S', \ldots$ for hyperformulae; the notation $S \mid S'$ means set union and $S \mid \alpha$ and $\alpha \mid S$ stand for $S \mid \{\alpha\}$ and $\{\alpha\} \mid S$, respectively.

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Hyperformulae and derivations

Definition

Let Γ be a set of propositional modal formulae and let K be a set of multiple-conclusion rules. A K-*hyperproof* (or a K-*derivation* or just a *derivation*) under assumptions Γ is a finite list of hyperformulae $S_1, \ldots, S_n$ such that each S_i in it matches one of the following requirements:

(i) S_i is of the kind $\alpha \mid S$, where $\alpha \in \Gamma$ or α is a tautology or α is an instance of the **K** distribution axiom;

(ii) S_i is obtained from hyperformulae preceding it by applying a rule from K or the necessitation rule or the modus ponens rule.

We write $\Gamma \vdash_K S$ to mean that there is a K-derivation ending with S.

Hyperformulae and derivations

An important remark is in order for (ii): when we say that S_i is obtained by applying an inference rule, *we include uniform substitution and weakening in the application of the rule*. Thus, if the rule is

$$\frac{\gamma_1, \quad \cdots, \quad \gamma_n}{\delta_1 \mid \cdots \mid \delta_m} \ (R)$$

when we say that S_i is obtained from (R), we mean that there are a hyperformula S and a substitution σ such that S_i is of the kind $S \mid \delta_1\sigma \mid \cdots \mid \delta_m\sigma$ and that there are $j_1, \dots, j_n < i$ such that S_{j_1} is of the kind $S \mid \gamma_1\sigma$, and ... and S_{j_n} is of the kind $S \mid \gamma_n\sigma$.

Hyperformulae and derivations

In other words, when rule (R) is used, we apply a substitution to its contextual form

$$\frac{\gamma_1 \mid S, \quad \dots, \; \gamma_n \mid S}{\delta_1 \mid \cdots \mid \delta_m \mid S} \; (R)$$

Proposition

We have $K \vdash \Gamma / S$ iff there is a K-derivation under assumptions Γ ending in S.

The Step embedding theorem (hyper version)

It is routine to define what it means for a modal calculus K (seen as a set of reduced multiconclusion rules) to enjoy fmp and bpp.
It is also routine to define validation of a reduced multiconclusion rule in a step algebra and in a step frame. We have

Theorem

Let K be a modal calculus. Then K enjoys both bpp and fmp iff every finite conservative step-frame validating K is a p-morphic image of a finite Kripke frame validating K.

As examples you can take the calculi obtained by translating hypersequent rules for **S4.3**, **S5** into multiconclusion rules (these systems axiomatize the class of corresponding *prime* algebras).

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Homomorphic Images and Stability

A *stable embedding* of a modal algebra $\mathfrak{A} = (A, \Diamond)$ into a modal algebra $\mathfrak{B} = (B, \Diamond)$ is an injective Boolean morphism $\mu : A \to B$ such that we have $\Diamond\mu(x) \leq \mu(\Diamond x)$ for all $x \in A$.

A class $\mathcal{C}$ of modal algebras is said to be *stable* iff whenever $\mathfrak{B} \in \mathcal{C}$ and $\mathfrak{A}$ has a stable embedding into $\mathfrak{B}$, then $\mathfrak{A} \in \mathcal{C}$ too.

We have dual notions for frames. $\mathfrak{F} = (W, R)$ is a *homomorphic image* of $\mathfrak{F}' = (W', R')$ iff there is a surjective map $f : W' \to W$ such that xRy implies $f(x)R'f(y)$ for all $x, y \in W'$ (in case $\mathfrak{F}, \mathfrak{F}'$ are descriptive, f is asked to be continuous too).

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A modal calculus K is *stable* iff so is the class of modal algebras validating it (equivalently: the class of descriptive frames validating it).

The following Theorem is proved in (N. & G. Bezhanishvili & Iemhoff 2014):

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(i) *A modal calculus K is stable iff it is axiomatizable via stable characteristic rules.*

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Stable Characteristic Rules

Stable characteristic rules are the rules associated with finite Kripke frames in the following way.

Let $\mathfrak{F} = (F, R_F)$ be a finite frame. For every $a \in F$ we introduce a new propositional variable x_a. The *modal stable rule* of $\mathfrak{F}$ is

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i \neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box \bigvee_{b \in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}} \quad (r_{\mathfrak{F}})$$

where we suppose that $F = \{a_1, \ldots, a_n\}$.

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The following proposition is proved in (N. & G. Bezhanishvili & Iemhoff 2014):

Proposition

Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra. Then

1. $\mathfrak{A}$ *does not validate* $(r_{\mathfrak{F}})$ *iff there is a stable embedding of* $\mathfrak{F}^*$ *into* $\mathfrak{A}$.
2. $\mathfrak{A}$ *does not validate* $(r_{\mathfrak{F}})$ *iff there is a surjective stable map from* $\mathfrak{A}_*$ *onto* $\mathfrak{F}$.

Bpp for Stable Calculi

To get bpp however we need to modify rules $(r_{\mathfrak{F}})$ as shown below.

$$\frac{\bigvee_{i=1}^{n} x_{a_i}, \quad \bigwedge_{i\neq j} \neg(x_{a_i} \wedge x_{a_j}), \quad \bigwedge_{i=1}^{n}(x_{a_i} \to \Box r_{a_i}), \quad \bigwedge_{i=1}^{n}(r_{a_i} \to \bigvee_{b\in R_F(a_i)} x_b)}{\neg x_{a_1} \mid \cdots \mid \neg x_{a_n}}$$

Lemma

Rules $(r_{\mathfrak{F}}^{+})$ and $(r_{\mathfrak{F}})$ are inter-derivable.

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Lemma

Rules $(r^{+}_{\mathfrak{F}})$ and $(r_{\mathfrak{F}})$ are inter-derivable.

Bpp for Stable Calculi

Theorem

Any modal calculus axiomatized by rules of the kind $(r^+_{\mathfrak{F}})$ *enjoys bpp and fmp.*

Corollary

Let $\mathcal{C}$ *be a stable class of (ordinary) Kripke frames such that membership of a finite frame to* $\mathcal{C}$ *is decidable. Then validity of a formula (more generally, of a rule) in* $\mathcal{C}$ *is decidable as well.*

Bpp for Stable Calculi

Theorem

Any modal calculus axiomatized by rules of the kind $(r^+_{\mathfrak{F}})$ enjoys bpp and fmp.

Corollary

Let $\mathcal{C}$ be a stable class of (ordinary) Kripke frames such that membership of a finite frame to $\mathcal{C}$ is decidable. Then validity of a formula (more generally, of a rule) in $\mathcal{C}$ is decidable as well.

Conclusions

- step methods seem to be quite effective in jointly proving bpp and fmp;
- in simple cases the application of the methods is fully automatic (in a sense we are *mechanizing the metatheory* of modal logic!);
- in more complex cases some ingenuity is needed, still uniform arguments often work;
- entire classes of logics can be covered (see the above results on stable classes);
- the scalability of the methods is to be tested for more complicated logics arising in computer science applications.

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